How Does a Vernier Scale Work?

by

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Pierre Vernier lived from 1584-1638 in what is now France, but at that time where he lived was a Spanish Hapsburgs possession. He held various government posts including Conseiller du Roi (Counselor to the King). He worked as an engineer but was a mathematician and scientist. His most famous invention was what we now call a Vernier caliper. However, Vernier scales can be found on many different kinds of devices that don't necessarily measure things linearly as does a caliper device. For instance, Vernier dials (round cylindrical scales) are commonly found on sextants where the main scale is also partly circular.

In any case, we can think that a Vernier device consists of two parallel scales, and when the device is in a linear form, both scales can be thought of as being laid out with marks similar to those found on a ruler. First, there is the Main Scale where each unit is usually divided into ten sub units or decimal units. The secondary scale is also marked with ten marks, but the difference is that those ten marks have a total length that is equivalent to only 9 of the tenth-units on the Main Scale. The secondary scale is called the Vernier scale and this special scale is used to accurately read units to the nearest $\frac{1}{100}$ of a unit on the Main Scale. More precisely, the Vernier scale actually measures hundredths of units on the Main Scale. We can best understand how the two scales are related to each other by showing a particular example.

Suppose we wish to use the Main Scale like a ruler to measure the length of the following black bar rectangle as is indicated in Figure 1.



Figure 1. Measuring the length of the black bar rectangle with only a Main Scale.

We line up the left edge of the solid black bar with the 0.0 mark on the Main Scale. When we try to read the length going to the right, we have to estimate the third significant digit. For example, we can easily read the length as 0.6+ units. The Main Scale is hard to read when it comes to reading the next digit of accuracy. For instance, it is hard to tell if the length of the black bar should be 0.65 or 0.66 or 0.67. Probably one of these three values is correct, but how can we decide which one is the closest to the true value? We can't realistically expect read the length any more accurately than to the nearest hundredth of a unit, but can we read it to that much accuracy? Vernier figured a way to squeeze one more accurate digit out of such a reading on the Main Scale.

What Vernier did was to align a secondary scale to the Main Scale. But his secondary scale was special in that it consists of 10 more marks which are spread out exactly evenly within the space of exactly 0.9 of a unit of length on the Main Scale. This means the exact spacing between the marks on the Vernier scale is 0.09 main units. Vernier scale marks are not labeled as decimals, in fact only three marks are labeled with the integers 0 and 5 and 10. As we will see, the Vernier marks are simply used as counters.

Now think of the multiples of 0.09. We show the multiples in Table 1 below.

Vernier Mark Count \rightarrow	0	1	2	3	4	5	6	7	8	9	10
corresponding decimal \rightarrow	0.00	0.09	0.18	0.27	0.36	0.45	0.54	0.63	0.72	0.81	0.90

Table 1. Comparison of Vernier marks with Main Scale values.

Next, we align the 0-marks on the left of the two scales. The Vernier scale is the lower scale as shown in Figure 2 below. The black bar is no longer shown because it is not relevant now.



Figure 2. The Vernier Scale matched up against the Main Scale.

Now the next thing to learn is that there will be at most two marks on the Vernier scale that most closely match one of the small decimal marks on the Main Scale. In Figure 2, it is obvious that the 10 mark on the Vernier fits exactly with the 0.90 mark on the Main Scale, as do the two marks at the zeros of both scales. Of course we have deliberately aligned the two scales at their zero marks. In Table 2 below we show a comparison of the values of the two scales.

Vernier Mark $\# \rightarrow$	0	1	2	3	4	5	6	7	8	9	10
Main Scale Reading \rightarrow	0.00	0.09	0.18	0.27	0.36	0.45	0.54	0.63	0.72	0.81	0.90
Error to closest mark \rightarrow	0.00	0.01	0.02	0.03	0.04	0.05	0.04	0.03	0.02	0.01	0.00

Table 2. Comparing the Vernier with the Main Scales when both are matched at zero.

Next, let's think of sliding the Vernier scale slightly to the right until its zero mark aligns at exactly the 0.13 mark on the Main Scale. Of course, we may not think we can read the Main Scale that accurately, but the next figure shows the values corresponding to each of the 10 Vernier marks. There is nothing special about having chosen the 0.13 position on the Main Scale, but we wanted to see where and how the 10 marks on Vernier compare with the Main Scale when the Vernier starting position is moved to an arbitrary point on the Main Scale. In an actual caliper device, the object being measured is aligned with some position on the Main Scale like 0.13. Thus the 0 mark on the Vernier can be used to read the first decimal place value. As we will learn, another mark on the Vernier scale determines the next decimal digit in the measurement. Study Figure 3 below.



Figure 3. The Vernier Scale with its zero mark aligned at 0.13 on the Main Scale.

This time we can write the values we see (yes the are difficult to see!) in Figure 3 above in the following table.

Vernier Mark $\# \rightarrow$	0	1	2	3	4	5	6	7	8	9	10
Main Scale Reading \rightarrow	0.13	0.22	0.31	0.40	0.49	0.58	0.67	0.76	0.85	0.94	1.03
Error to closest mark \rightarrow	0.03	0.02	0.01	match	0.01	0.02	0.03	0.04	0.05	0.04	0.03
Main Scale closest mark \rightarrow	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	?	0.90	1.10

Table 3. The values and errors associated with the scales shown in Figure 3.

This time there is exactly one Vernier Scale mark that matches exactly with one of the decimal marks on the Main Scale. Now the secret to reading a Vernier Scale is to read the next decimal place in the measurement by counting the position of the matching Vernier mark. In Table 3 we can see the mark is the 3rd Vernier mark. Thus when we read the measurement using the Vernier Scale, we first accurately read 0.1 on the Main Scale, and then we more accurately read this as 0.13 because the 3rd mark on the Vernier determines the hundredths place value.

Now you need to think why the results in Table 3 will always typically happen. That is, when the object being measured has a difficult-to-read partial value at the 0 Vernier mark on the Main Scale, then one of the Vernier's ten marks will align with one of the Main Scale marks. As yet another example, consider Figure 4 below in which the Vernier scale zero mark is at 0.25 on the Main Scale.



Figure 4. The Vernier Scale with its zero mark aligned at 0.25 on the Main Scale.

Table 4 below can be used to analyze the significant marks on both scales in Figure 4.

Vernier Mark $\# \rightarrow$	0	1	2	3	4	5	6	7	8	9	10
Main Scale Reading \rightarrow	0.25	0.34	0.43	0.52	0.61	0.70	0.79	0.88	0.97	1.06	1.15
Error to closest mark \rightarrow	0.05	0.04	0.03	0.02	0.01	match	0.09	0.02	0.03	0.04	0.05
Main Scale closest mark \rightarrow	?	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10	?

Table 4. The values and errors associated with the scales shown in Figure 4.

In this case it is the 5th Vernier mark that matches the 0.7 Main Scale mark. Note that the value of the Main Scale mark is NOT the relevant value. By this we mean the value of 0.7 is irrelevant to the correct value to be read from the scale, but the 5th Vernier mark means the reading measurement should be 0.25. The 0.2 value can be read off the Main Scale while the next digit 5 is because the 5th Vernier mark is the matching mark.

Just so you can see another example of a different type of Vernier scale, Figure 5 below shows a reading from an actual sextant machine.



Figure 5. A sextant machine showing an angle of 29 degrees and 42.5 minutes.

Now you are not expected to know what a sextant machine looks like, nor what it does, but you can see three different scales in the picture in Figure 5 above. Begin by looking at the right side of the figure, halfway up. You can see a scale that starts at 0 and if you move from right to left along the large curved scale you can see that the angle scale goes beyond 80 degrees near the upper-left corner of the figure. This is the angle-in-degrees scale of a sextant. The arrow near the very middle of the figure, near the number 30 on the large curved scale, is used to read the angle to the nearest degree. In this case the arrow tip is just slightly to the right of 30 on the angle scale so the angle to the nearest degree is properly read as 29+ degrees.

Next, there are two round dials with numbers in the lower left part of the figure. The larger round dial is the minutes scale for the sextant. We can think of this larger dial as the Main Scale. The smaller round dial adjacent and immediately to the right of the larger dial is the Vernier Scale. Now look at the 0-mark on the smaller round dial Vernier. Read the value on the larger dial across from the 0-mark. Your reading should be somewhere between 42 and 43. Then note that the Vernier scale has numbered marks on it, and mark 5 on the Vernier just happens to match the main dial value of 47, while all the other Vernier marks don't match anything on the Main Scale dial. We determine the true value of the 0-mark Vernier point is supposed to be 42.5 minutes of an angle. Thus the reading for the entire sextant shown in Figure 5 above is the angle that is 29°42.5'.

In this paper we have not actually shown nor explained how to read either an inside or an outside measurement with an actual caliper device. Our purpose was not to explain how a caliper works, but rather to discuss the mathematical aspects of a Vernier scale. We will finish by giving a statement and proof of the mathematical principle behind the Vernier scale. We state the following theorem in terms of integers because all of the decimal values given so far in this paper could be turned into whole numbers by multiplying everything by 100.

- **Theorem.** Given any positive integer n where $n \mod 10 \neq 0$, there exists a unique integer k such that $(n+9k) \mod 10 = 0$, where $1 \le k \le 9$.
 - Proof: Let $k = n \mod 10$. Then since $n \mod 10 \neq 0$ we know k must be an integer in the range between 1 and 9. Thus there exists another integer q such that n = 10q + k. Then we have:

$$n + 9k = 10q + k + 9k = 10q + 10k = 10(q + k)$$

This equation shows that n + 9k is a multiple of 10, which means $(n + 9k) \mod 10 = 0$. To establish the uniqueness of k, suppose i is an integer between 1 and 9 such that n + 9i is also a multiple of 10. Assume n + 9i = 10p. Then n + 9k and n + 9i are both multiples of 10 and thus their difference is also a multiple of 10. This means (n + 9k) - (n + 9i) = 9k - 9i = 9(k - i) = 10s for some integer s. Now the only first (smallest) multiples of 9 that are also multiples of 10 are -90, or 0 or 90. Thus either 9(k - i) = 0 or $9(k - i) = \pm 90$. This later equation would imply $k - i = \pm 10$ and this is impossible because the two inequalities $1 \le k \le 9$ and $1 \le i \le 9$ imply $-8 \le k - i \le 8$. Thus we must have 9(k - i) = 0 which implies k - i = 0 which means k = i. The integer k as described must be unique. A consequence of the previous theorem is that exactly one of the Vernier marks between 1 and 9 must match exactly one of the exact decimal marks on the Main Scale when the 0-mark on the Vernier scale is not already lined up at one of the exact decimal marks on the Main Scale. Compare Figures 3 and 2.

In the rare case like Figure 2 where the 0-mark on the Vernier is lined up with an exact decimal mark on the Main Scale, then the 10-mark on the Vernier will also be a second matching Vernier mark. In this rare case $n \mod 10 = 0$. Otherwise, only one of the nonzero and non-10 Vernier marks will align with one of the decimal marks, and in that case $(n \mod 10 \neq 0)$ we can uniquely and precisely decide how to read the scale by using a further decimal place between 1 and 9. Such a reading is accurate to $\frac{1}{10}$ of the distance between the $\frac{1}{10}$ decimal marks making the reading accurate to the nearest $\frac{1}{100}$ of a unit on the Main Scale. The secondary Vernier scale is a very ingenious little device!

As an example of another device that uses a Vernier scale on a round cylinder, Figure 6 below shows a micrometer. If you look carefully you can see about the first five marks on the Vernier scale that has horizontally marked lines. The Main Scale is to the right of the Vernier Scale. A third scale that appears in the following figure is below the Vernier scale and is used to read the first two decimals of the gap distance.



Figure 6. A micrometer device with a cylindrical Vernier scale.

This micometer can measure anything less than an inch long to the nearest ten thousandth of an inch. In this figure we can start reading the gap as 0.15+ inches. The main scale reading adds another digit 9 because the 0-mark of the Vernier scale is at just under the 10 mark on the main scale. This means the distance is 0.159+ inches without reading the matching Vernier mark. Finally, we add yet another digit from the Vernier scale which is impossible to see in the above figure, but we assume the matching mark

is the 8 counter on the Vernier scale. Thus the complete micrometer distance in the above figure could be read as 0.1598 inches.

Most modern caliper devices have either a dial scale or a digital readout and both dials and the digital readouts have replaced the Vernier Scales that are primarily found on older caliper devices. Figure 7 below is of a caliper with a digital readout that could be used to measure both the inside and outside dimensions of say a piece of pipe.



Figure 7. A modern caliper device with a digital readout.

The main consequence of having digital readouts is that anyone without any knowledge of how to read Vernier scales can make accurate measurements. Perhaps this situation isn't much different from kids who know how to read a digital watch but can't read an analog clock. And of course there will always be kids who don't know how to tie their own shoes but who do just fine with Velcro straps! Does anybody really know what time it is? Does anybody really care?